EXERCISESET 7, TOPOLOGY IN PHYSICS

- The hand-in exercise is the exercise 2.
- Please hand it in electronically at `topologyinphysics2019@gmail.com` (1 pdf!)
- Deadline is Wednesday April 3, 23.59.
- Please make sure your name and the week number are present in the file name.

Exercise 1: The goal of this exercise is get more familiar with the "global point of view" on connections. We start out with an abelian example with structure group $G = U(1)$. Recall that $SU(2) \cong S^3$ which becomes clear if we parameterize $SU(2)$-matrices as

$$\begin{pmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{pmatrix}, \quad z_0, z_1 \in \mathbb{C}, \ |z_0|^2 + |z_1|^2 = 1.$$

The group $SU(2)$ contains $U(1)$ as the diagonal matrices of the form

$$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \quad \varphi \in [0, 2\pi).$$

We now let $U(1)$ act on $SU(2)$ from the right.

a) Show that the quotient $SU(2)/U(1) \cong S^2$.

b) Show that the projection of the Maurer–Cartan form of $SU(2)$

$$g^{-1}dg = \begin{pmatrix} \bar{z}_0 dz_0 + \bar{z}_1 dz_1 & - \\ - & - \end{pmatrix}$$

onto the upper-left corner defines a $U(1)$-connection:

$$A = \bar{z}_0 dz_0 + \bar{z}_1 dz_1$$

c) Introduce the coordinates (z, φ) on S^3 by

$$\begin{aligned} z_0 &= re^{i\varphi} \\ z_1 &= rze^{i\varphi} \end{aligned} \qquad r^2 = \frac{1}{1+|z|^2}.$$

(Remark that z can be viewed as local coordinate on the quotient.) Show that the curvature is given by

$$F = i\frac{d\bar{z} \wedge dz}{(1+|z|^2)^2}$$

Evaluate $\int_{S^2} F$, knowing that the area of S^2 evaluated w.r.t. to the standard volume form $\frac{2i}{(1+|z|^2)2} dz \wedge d\bar{z}$ is 4π. Compare with Theorem 4.3 of the notes.

This set-up generalizes if we use the *quaternions* instead of the complex numbers. Recall that the quaternions $\mathbb{H}$ are just $\mathbb{R}^4$ with basis $\{1, i, j, k\}$, equipped with multiplication defined by

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = -k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Notice that this multiplication is *not* commutative!. We write $q = x_0 + x_1 i + x_2 j + x_3 k$ for a quaternion $v \in \mathbb{H}$, and denote by $\bar{q} = x_0 - x_1 i - x_2 j - x_3 k$ its conjugate. The equation

$$|q|^2 := q\bar{q} = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1,$$

defining the *unit quaternions*, then describes the 3-sphere in $\mathbb{R}^4$. From now on, we write u for a unit quaternion.

d) Show that the unit quaternions form a group isomorphic to $SU(2)$.
e) We now form the group G:

$$\begin{pmatrix} q_0 & -\bar{q}_1 \\ q_1 & \bar{q}_0 \end{pmatrix}, \quad q_0, q_1 \in \mathbb{H}, \ |q_0|^2 + |q_1|^2 = 1,$$

and $SU(2)$ act from the right via the diagonal embedding

$$\begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}, \quad u \text{ a unit quaternion.}$$

Show that the quotient $G/SU(2) \cong S^4$.
f) Similar computations as in b) show that

$$A = \bar{q}_0 dq_0 + \bar{q}_1 dq_1$$

defines an $SU(2)$-connection. Introduce local coordinates (q, u) on G by

$$\begin{aligned} z_0 &= \rho u \\ z_1 &= \rho q u \end{aligned} \qquad \rho^2 = \frac{1}{1 + |q|^2}.$$

Show that the curvature $F = dA + A \wedge A$ can be written as

$$uFu^{-1} = \frac{d\bar{q} \wedge dq}{(1 + |q|^2)^2}$$

g) The metric on S^4 is written in the local coordinate q as

$$ds^2 = r^4 d\bar{q} dq.$$

Check that the connection A above is (anti-)selfdual. Compute $\int \mathrm{Tr}(F \wedge F)$ by comparing the integrand to the volume form on S^4.

⋆ **Exercise 2: The Chern character.**

a) Show that the restriction of an invariant symmetric polynomial

$$P : \mathrm{Mat}_r(\mathbb{C}) \times \ldots \times \mathrm{Mat}_r(\mathbb{C}) \to \mathbb{C}$$

to the subset diagonalizable matrices, defines a symmetric polynomial[1] in r variables. With this, relate the Chern classes to the elementary symmetric functions

$$\sigma_k(\lambda_1, \ldots, \lambda_r) := \sum_{1 \le i_1 \le \ldots \le i_k \le r} \lambda_{i_1} \cdots \lambda_{i_r}, \quad k = 1, \ldots, r.$$

b) It is an algebraic fact that any symmetric polynomial can be written as a linear sum of products of the σ_k's. Show that any characteristic class obtained from the Chern–Weil formalism can be expressed in terms of Chern classes by using this fact, the splitting principle and the results of a).

c) Consider the polynomial functions P_k on $\mathrm{Mat}_r(\mathbb{C})$ ($r \times r$-matrices) defined by the expansion

$$\mathrm{Tr}(e^{tA}) = P_0(A) + tP_1(A) + t^2 P_2(A) + \ldots, \quad A \in Mat_r(\mathbb{C}).$$

Show that the P_k are invariant, and therefore define characteristic classes $\mathrm{ch}_k(E) \in H^{2k}_{\mathrm{dR}}(M)$ of a vector bundle $E \to M$. Express ch_1 and ch_2 in terms of Chern classes. Have you seen ch_2 before?

d) The *Chern character* is defined as the sum

$$\mathrm{ch}(E) := \sum_{k \ge 0} \mathrm{ch}_k(E) \in H^{\bullet}_{\mathrm{dR}}(M).$$

Why is this a finite sum? Show that the Chern character satisfies $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$.

Exercise 3: The Chern Simons form.

a) If you view the trace on matrices as an invariant polynomial via $(A, B) \mapsto \mathrm{Tr}(AB)$, what is the associated characteristic class?

b) Explain why the characteristic class in a) is, up to a normalization of $4\pi^2$, an integral cohomology class.

c) Suppose that $E \to M$ is a trivial vector bundle and let $\nabla = d + A$ be a connection. Show that the transgression form $L(d, d + A)$ is exactly the Chern–Simons form $\mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$ discussed before.

[1] A symmetric polynomial is a polynomial function Q of r-variable that is invariant under permutations of the variables: $Q(\lambda_1, \ldots, \lambda_r) = Q(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(r)})$, $\forall \tau \in S_r$.

Exercise 4: The Euler Class. In this exercise we will consider oriented real vector bundles (and in the end also complex ones). One way to characterize an orientation on a real rank n vector bundle $\pi\colon E \to M$ over the manifold M is to say that relative to some open cover $\{U_\alpha\}$ we may pick the transition functions denoted by $\phi_{\alpha\beta}\colon U_{\alpha\beta} \to GL(n, \mathbb{R})$ to have values in the subgroup $SO(n)$ of $GL(n, \mathbb{R})$. So we will now assume that $\phi_{\alpha\beta}\colon U_{\alpha\beta} \to SO(n)$, i.e. we will consider oriented vector bundles. This means in particular that we can choose point-wise linearly independent sections $e_{\alpha,i}\colon U_\alpha \to E|_{U_\alpha}$ for $i = 1, \ldots, n$ for any α such that $\phi_{\alpha\beta}(x)^j_i e_{\beta,j}(x) = e_{\alpha,i}(x)$ for all $x \in U_\alpha \cap U_\beta$. Given $x \in M$ and any α such that $x \in U_\alpha$ we can now consider polar coordinates $(r_\alpha, \theta_{\alpha,1}, \ldots, \theta_{\alpha,n-1})$ on $E_x \backslash \{0\}$ by treating the $e_{\alpha,i}$ as standard coordinates. Note also that this system of coordinates varies smoothly with x.

i) Show that, if $x \in U_{\alpha\beta}$, then $r_\alpha(x) = r_\beta(x)$.

For simplicity's sake let us set $n = 2$ from now on.

ii) Argue that we may pick functions $\varphi_{\alpha\beta} \in C^\infty(U_{\alpha\beta})$ such that

$$\theta_\beta = \theta_\alpha + \pi^* \varphi_{\alpha\beta}$$

holds on $E|_{U_{\alpha\beta}} \backslash 0(U_{\alpha\beta})$ for 0 the zero section.

We note that

$$d\varphi_{\alpha\beta} - d\varphi_{\alpha\gamma} + d\varphi_{\beta\gamma} = 0$$

on the triple intersections $U_{\alpha\beta\gamma}$ (as a bonus exercise prove this).

iv) Show that there exist one-forms $\xi_\alpha \in \Omega^1(U_\alpha)$ such that

$$\frac{1}{2\pi} d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha.$$

HINT: consider $\frac{1}{2\pi} \sum_\gamma \rho_\gamma \varphi_{\alpha\gamma}$ for $\{\rho_\gamma\}$ a partition of unity subordinate to $\{U_\gamma\}$.

v) Show that the two-forms $d\xi_\alpha$ define a class $e(E) \in H^2_{\mathrm{dR}}(M)$ that is independent of the choice of $\xi'_\alpha s$.

The class $e(E)$ is called the Euler class of the oriented vector bundle E. Given a rank 1 complex vector bundle V it is given by transition functions $\phi_{\alpha\beta}$ with values in $U(1)$ and by considering the isomorphism $SO(2) \simeq U(1)$ given by

$$e^{i\phi} \mapsto \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

it gives rise to a rank 2 oriented real vector bundle $V_\mathbb{R}$.

vii) Show that $e(V_\mathbb{R}) = c_1(V)$.

Finally let us compute such a class. Consider the two-sphere S^2 and note that it is isomorphic to the complex manifold $\mathbb{P}^1$. This means that the tangent bundle $TS^2 = V_\mathbb{R}$ for V a rank 1 complex bundle.

viii) Show that

$$\int_{S^2} e(TS^2) = \chi(S^2)$$

where $\chi(M) := \sum_{i=0}^{\infty}(-1)^i \mathrm{Dim}\ H^i_{\mathrm{dR}}(M)$ denotes the so-called *Euler characteristic*.

HINT: Split the integral into a sum of integrals over the north and south hemispheres (keep orientation in mind). Consider the usual cover of S^2 given by U_N and U_S by deleting the south and north poles respectively. To determine φ_{NS} consider orthonormal vector fields e^1_N, e^2_N on U_N and e^1_S, e^2_S on U_S for the usual Riemannian metric.